

# Asymptotic behavior in the scalar field theory

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**Abstract.** An asymptotic solution of the system of Schwinger-Dyson equations for four-dimensional Euclidean scalar field theory with interaction  $\frac{\lambda}{2}(\phi^*\phi)^2$  is obtained. For  $\lambda > \lambda_{cr} = 16\pi^2$  the two-particle amplitude has the pathology-free asymptotic behavior at large momenta. For  $\lambda < \lambda_{cr}$  the amplitude possesses Landau-type singularity.

PACS number: 11.10.Jj.

## 1 Introduction

The problem of asymptotic behavior at large momenta (or, at short distances) is one of oldest problem of quantum field theory. In general, this problem has been solved only for the asymptotically-free models. A solution of the problem of asymptotic behavior for other strictly renormalized models – quantum electrodynamics, self-interacting scalar field, Yukawa interaction – requires to go out the framework of weak-coupling approximation.

The first attempt to define the asymptotic behavior in quantum electrodynamics was made by Landau and coworkers in the 1950s [1]. The result was unrehearsed: the photon propagator included the non-physical pole in Euclidean region of momenta. Then similar poles were indicated in other strictly renormalized models: self-interacting scalar field and Yukawa interaction. The presence of such singularities in Euclidean region violates general principles of the local field theory and is a serious problem for these models.

A method of removing these non-physical singularities was proposed soon in works [2]. This method is based on the application of Kallen–Lehmann representation to restore the correct analytical structure and remains the basic tool to solve the problem of the non-physical singularities to our time. However, the absence of dynamical foundation enforced to search a more detailed investigation for the role of these singularities.

Further development of the quantum field theory has demonstrated that these non-physical singularities arise practically inevitably in the framework of any known non-perturbative methods: at the renormalization-group summation, in the frameworks of  $1/N$ -expansion and mean-field expansion, etc.<sup>2</sup>

A widespread opinion is formulated as a mathematical inconsistency of the quantum field models that are not asymptotically free. There is rigorous theorem

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<sup>2</sup>See review [3] for the historical survey and further references.

[4] that the four-dimensional scalar field theory with  $\phi^4$  interaction on the lattice does not have an interacting continuum theory as its limit for zero lattice spacing, i.e. the theory is trivial. Weinberg argues [5], however, that this argument is inconclusive due to an uncertainty of the continuous limit in this model.

Today the situation with triviality of  $\phi^4$  theory is vague as before. Recent papers (see, e.g., [6], [7], [8]) in this topic maintain incompatible statements. While works [7] confirm the triviality scenario, works [6], [8] state non-trivial behavior in the strong-coupling limit.

The asymptotic short-distance region in the models without asymptotic freedom is the region of strong (or, more exactly, non-weak) coupling, and it is the main difficulty in its investigation. It seems that the above-mentioned standard methods are too tethered to the weak-coupling region and are not enough meaningful to inform us about behavior at short distances in these models. In this paper the new approximation for the  $\phi^4$  theory is investigated. This approximation is based on a system of Schwinger-Dyson equations (SDEs) for the propagator and the two-particle function. The first equation is the exact SDE for the propagator, and the second equation is a truncated SDE for the two-particle function ("two-particle approximation"). This approximation is considered as a first non-trivial step of the sequence of general  $n$ -particle approximations, which tends to the exact infinite system of SDEs at  $n \rightarrow \infty$ . An investigation of some truncation of SDEs is not a news, of course. A new step is the consideration of a system of these equations instead of the usual consideration of some single truncated equation.

A structure of the paper is as follows: in section 2, the necessary notations and definitions are given; SDE for the generating functional of Schwinger functions is introduced in the formalism of a bilocal source. Using of the bilocal source is an essential point of the construction. We consider using this presumably as a convenient choice of the functional variable. In particular, this variable is very convenient for the construction of the mean-field expansion (MFE) by method of work [11], which is presented in section 2. The existence of the Landau pole in the two-particle amplitude of the leading approximation of MFE is also demonstrated in this section.

In section 3, a general construction of the approximation scheme for the system of SDEs is given. In section 4, an alternative view to the approximation of preceding section is given. The system of equations for the propagator and the two-particle function is considered as a basic approximation for the construction of the modification of MFE of section 2. In section 5, the renormalization of the system of equations is made and some supplement simplifications are discussed.

In section 6, the asymptotic solution of the equation for the two-particle amplitude at large momenta is presented. In section 7, the asymptotic behavior of the amplitude at large momenta is discussed. The amplitude in this model possesses a self-consistent behavior (as a constant plus a decreasing oscillating term) at the values of the renormalized coupling  $\lambda \in (\lambda_{cr}, 2\lambda_{cr})$ , where  $\lambda_{cr} = 16\pi^2$ . At  $\lambda < \lambda_{cr}$  the amplitude has some Landau-type singularity in the pre-asymptotic region. At  $\lambda > 2\lambda_{cr}$  the method of solution is, strictly speaking, cannot be applied due to the

negative value of the field-renormalization constant, but if one assumes formally such values, the results of the permitted interval can be continued to the region of strong coupling. Conclusions are presented in section 8.

## 2 Preliminaries. SDE for the generating functional, mean-field expansion and Landau pole

Consider the theory of a complex scalar field  $\phi(x)$  in a four-dimensional Euclidean space ( $x \in E_4$ ) with the Lagrangian

$$\mathcal{L} = -\partial_\mu \phi^* \partial_\mu \phi - m_0^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2 \quad (1)$$

in the symmetric phase ( $m_0^2 > 0$ ,  $\lambda > 0$ ). The generating functional of  $2n$ -point ( $n$ -particle) Schwinger functions can be written as a functional integral

$$G = \int D(\phi, \phi^*) \exp \left\{ \int dx \mathcal{L}(x) - \int dx dy \phi^*(x) \eta(x, y) \phi(y) \right\}, \quad (2)$$

where  $\eta(x, y)$  is a bilocal source. The  $n$ th derivative of  $G$  over  $\eta$  with the source being switched off is the  $n$ -particle Schwinger function. The propagator of the field  $\phi$  is

$$\Delta(x - y) = \langle \phi(x) \phi^*(y) \rangle = - \frac{\delta G}{\delta \eta(y, x)} \Big|_{\eta=0},$$

the two-particle function is the second derivative of  $G$ , etc.<sup>3</sup>

The translational invariance of the functional integration measure leads to relation

$$\int D(\phi, \phi^*) \frac{\delta}{\delta \phi^*(x)} \phi^*(y) \exp \left\{ \int dx \mathcal{L}(x) - \int dx dy \phi^*(x) \eta(x, y) \phi(y) \right\} = 0,$$

which can be rewritten as the functional-differential SDE for generating functional  $G$ :

$$(m_0^2 - \partial_x^2) \frac{\delta G}{\delta \eta(y, x)} + \int dy_1 \eta(x, y_1) \frac{\delta G}{\delta \eta(y, y_1)} + \delta(x - y) G = \lambda \frac{\delta^2 G}{\delta \eta(x, x) \delta \eta(y, x)}. \quad (3)$$

To construct the mean-field expansion for the generating functional we consider as a leading approximation the following equation

$$\lambda \frac{\delta^2 G^{(0)}}{\delta \eta(x, x) \delta \eta(y, x)} - (m_0^2 - \partial_x^2) \frac{\delta G^{(0)}}{\delta \eta(y, x)} - \delta(x - y) G^{(0)} = 0, \quad (4)$$

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<sup>3</sup>A formalism of the bilocal source in the quantum field theory was first elaborated by Dahmen and Jona-Lasinio [9]. For further development and references see, e.g., [10]

whose solution is (in operator notations)

$$G^{(0)} = \exp\{-\Delta_0 \cdot \eta\},$$

where  $\Delta_0(p) = (m^2 + p^2)^{-1}$  and  $m^2 = m_0^2 + \lambda\Delta_0(x=0)$ . Here  $\Delta_0(x=0) \equiv \int_{\Lambda} \frac{d^4 p}{(2\pi)^4} \Delta_0(p)$ , and label  $\Lambda$  means that some regularization implied.

The leading approximation generates linear iteration scheme for SDE (3)

$$G = G^{(0)} + G^{(1)} + \dots + G^{(n)} + \dots,$$

where

$$\lambda \frac{\delta^2 G^{(n)}}{\delta\eta(x, x)\delta\eta(y, x)} - (m_0^2 - \partial_x^2) \frac{\delta G^{(n)}}{\delta\eta(y, x)} - \delta(x-y)G^{(n)} = \int dy_1 \eta(x, y_1) \frac{\delta G^{(n-1)}}{\delta\eta(y, y_1)}. \quad (5)$$

An idea of this iterative scheme is as follows: we shall consider "equation with constant coefficients" (4) as a leading approximation, i.e., equation (3) with the second term omitted. This term contains the source  $\eta$  manifestly. The Schwinger functions are the derivatives of  $G(\eta)$  in zero and only the behavior of  $G$  near  $\eta=0$  is essential, therefore such an approximation seems to be acceptable. The term omitted contains the source and should be treated as a perturbation.

The general solution of equation (5) is the functional  $G^{(n)} = P_{2n}G^{(0)}$ , where  $P_{2n}$  is a polynomial of  $2n$ th degree on source  $\eta$ . Therefore at the  $n$ th step the computation of Schwinger functions reduces to solving a closed system of  $2n$  linear integral equations.

The first-step generating functional  $G^{(1)}$  reads

$$G^{(1)} = \left( \frac{1}{2} G_2 \cdot \eta^2 - \Delta_1 \cdot \eta \right) G^{(0)}.$$

Here  $G_2$  is the leading-order two-particle function and  $\Delta_1$  is the next-to-the-leading-order correction to the propagator.

The iteration equation at  $n=1$  gives the equation for the two-particle function

$$(m^2 - \partial_x^2) G_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \delta(x-y') \Delta_0(x'-y) - \lambda \Delta_0(x-y) G_2 \begin{pmatrix} x & x \\ x' & y' \end{pmatrix}, \quad (6)$$

whose solution is

$$G_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \Delta_0(x-y') \Delta_0(x'-y) - \int dx_1 dx_2 \Delta_0(x-x_1) \Delta_0(x'-x_2) f(x_1-x_2) \Delta_0(x_1-y) \Delta_0(x_2-y'), \quad (7)$$

where in the momentum space

$$f(p) = \frac{\lambda}{1 + \lambda L_0(p)}, \quad (8)$$

and

$$L_0(p) = \int_{\Lambda} \frac{d^4 q}{(2\pi)^4} \Delta_0(p+q) \Delta_0(q) \quad (9)$$

is the single scalar loop.

Note, that first term in formula (7) is the missed disconnected part of the two-particle function of a leading approximation. Hence, the connected structure of the two-particle function is restored at the first step of iterations. Such a peculiarity of the iteration scheme is originated by the bilocal source and is not something exceptional: as is well-known, the similar phenomenon appears also in constructing  $1/N$ -expansion in the bilocal source formalism. The crossing properties of two-particle function in such iteration schemes are also restored stage-by-stage when the next terms in the expansion are taken into account (a discussion of this topic see in work [12]).

All above formulae for the propagator  $\Delta_0$  and the amplitude  $f$  contain divergent integrals and should be renormalized. In correspondence with the standard recipe we introduce the renormalized Lagrangian

$$\mathcal{L} = -\partial_{\mu} \phi^* \partial_{\mu} \phi - m^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2, \quad (10)$$

where  $\phi, m$  and  $\lambda$  are the renormalized field, mass and coupling, and add the counter-terms

$$\Delta \mathcal{L} = -(z-1) \partial_{\mu} \phi^* \partial_{\mu} \phi - \delta m^2 \phi^* \phi - (z_{\lambda}-1) \frac{\lambda}{2} (\phi^* \phi)^2, \quad (11)$$

which absorb the divergences.

Full Lagrangian  $\mathcal{L}_b = \mathcal{L} + \Delta \mathcal{L}$  can be written as

$$\mathcal{L}_b = -\partial_{\mu} \phi_b^* \partial_{\mu} \phi_b - m_b^2 \phi_b^* \phi_b - \frac{\lambda_b}{2} (\phi_b^* \phi_b)^2, \quad (12)$$

where

$$\phi_b = \sqrt{z} \phi, \quad m_b^2 = \frac{m^2 + \delta m^2}{z}; \quad \lambda_b = \frac{\lambda z_{\lambda}}{z^2}. \quad (13)$$

Then all above calculations are reproduced with Lagrangian (12), i.e., with the replacement  $m_0^2 \rightarrow m_b^2$ ,  $\lambda \rightarrow \lambda_b$ ,  $\Delta_0 \rightarrow \Delta_b$ ,  $G_2 \rightarrow G_{2b}$ ,  $f \rightarrow f_b$  etc., and the normalization conditions are imposed on the renormalized propagator  $\Delta$  and amplitude  $f$ . For the easement of the following calculations we choose the normalization point at zero momenta.

The normalization conditions for the propagator  $\Delta(p^2) = z^{-1} \Delta_b(p^2)$  are

$$\Delta^{-1}(0) = m^2, \quad (14)$$

$$\frac{d}{dp^2} \Delta^{-1}|_{p^2=0} = 1. \quad (15)$$

These conditions define the mass-renormalization counter-term  $\delta m^2$  and the field-renormalization constant  $z$ . It is easy to see that  $z = 1$  in the case, and, consequently,  $\Delta^{-1} = \Delta_b^{-1} = m^2 + p^2$ ,  $G_2 = G_{2b}$ . Thus, the amplitude  $f = f_b$  is

$$f(p) = \frac{\lambda z_\lambda}{1 + \lambda z_\lambda L_0(p)}. \quad (16)$$

The normalization condition for the amplitude is

$$f(0) = \lambda. \quad (17)$$

This condition defines the coupling-renormalization constant  $z_\lambda$  and the renormalized amplitude

$$f(p) = \frac{\lambda}{1 + \lambda L_r(p)}, \quad (18)$$

where

$$L_r(p) = L_0(p) - L_0(0) = -\frac{1}{16\pi^2} \int_0^1 dz \log(1 + z(1-z)\frac{p^2}{m^2}) \quad (19)$$

is the renormalized loop.

As it follows from equations (18) and (19) the renormalized amplitude  $f$  possesses a non-physical singularity (Landau pole) in the point  $p^2 = M_L^2$ , where  $M_L^2$  is a solution of the equation

$$1 + \lambda L_r(M_L^2) = 0.$$

This equation has a solution at any positive  $\lambda$  since function  $L_r$  covers all values from 0 to  $-\infty$ . At  $p^2 \rightarrow \infty$

$$L_r \simeq -\frac{1}{16\pi^2} \log \frac{p^2}{m^2}$$

and  $M_L \cong m \exp\{\frac{8\pi^2}{\lambda}\}$ . As was yet noted in the introduction, the same Landau pole arises at the calculations of renormalized amplitude by other methods: in the frameworks of  $1/N$ -expansion and renormalization-group summation.

### 3 The system of SDEs and two-particle approximation

From SDE (3) one can go to the functional  $Z = \log G$ . The SDE for  $Z$  is

$$(m_0^2 - \partial_x^2) \frac{\delta Z}{\delta \eta(y, x)} + \int dy_1 \eta(x, y_1) \frac{\delta Z}{\delta \eta(y, y_1)} + \delta(x-y) = \lambda \left[ \frac{\delta^2 Z}{\delta \eta(x, x) \delta \eta(y, x)} + \frac{\delta Z}{\delta \eta(x, x)} \frac{\delta Z}{\delta \eta(y, x)} \right] \quad (20)$$

The system of SDEs is an infinite set of equations for  $n$ -particle functions  $Z_n \equiv \delta^n Z / \delta \eta^n|_{\eta=0}$ . The first SDE is simply equation (20) with the source being switched off:

$$(m^2 - \partial_x^2)\Delta(x-y) = \delta(x-y) - \lambda Z_2 \begin{pmatrix} x & y \\ x & x \end{pmatrix}, \quad (21)$$

where  $m^2 = m_0^2 + \lambda\Delta(x=0)$  and  $\Delta \equiv -\delta Z / \delta \eta|_{\eta=0}$  is the propagator (or, the one-particle function). The second SDE is the derivative of (20) with the source being switched off:

$$(m^2 - \partial_x^2)Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \delta(x-y')\Delta(x'-y) - \lambda\Delta(x-y)Z_2 \begin{pmatrix} x & x \\ x' & y' \end{pmatrix} + \lambda Z_3 \begin{pmatrix} x & y \\ x & x \\ x' & y' \end{pmatrix}, \quad (22)$$

and so on. The  $n$ th SDE is the  $(n-1)$ th derivative of SDE (20) with the source being switched off and includes a set of functions from one-particle function  $\Delta$  to  $(n+1)$ -particle function  $Z_{n+1}$ . We call "the  $n$ -particle approximation of the system of SDEs" the system of  $n$  SDEs in which the first  $n-1$  equations are exact and  $n$ th SDE is truncated by omitting the  $(n+1)$ -particle function. It is evident that the sequence of such approximations goes to the exact set of SDEs at  $n \rightarrow \infty$ . The one-particle approximation is simply equation (21) without  $Z_2$ . This approximation has a trivial solution which is a free propagator. The two-particle approximation is a system of equation (21) and equation (22) without  $Z_3$ :

$$(m^2 - \partial_x^2)Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \delta(x-y')\Delta(x'-y) - \lambda\Delta(x-y)Z_2 \begin{pmatrix} x & x \\ x' & y' \end{pmatrix}, \quad (23)$$

which includes  $\Delta$  and two-particle function  $Z_2$ . This non-linear system will be the object of present investigation.

The idea of the present approximation scheme is very simple and natural. However, the calculations became more and more complicated at each following stage: e.g., the three-particle approximation is a system of three non-linear equations for the propagator, the two-particle function and the three-particle function. In following section some other scheme will be considered, which leads to same equations (21) and (23), but is more realistic in the calculational sense.

## 4 The particular solution of SDE for the generating functional and modified mean-field expansion

Another view to the origin of system (21) and (23) is based on a modification of the mean-field expansion of section 2 with taking into account a particular solution of

functional-derivative equation (20). It is easy to see that SDE (20) has the simple particular solution

$$Z_{part}(\eta) = \frac{1}{2} Z_2 \cdot \eta^2 - \Delta \cdot \eta, \quad (24)$$

where functions  $Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}$  and  $\Delta(x-y)$  satisfy to the system of equations (21) and (23).

This system should be supplemented by one more equation for  $Z_2$ :

$$\begin{aligned} \lambda Z_2 \begin{pmatrix} x & x \\ x' & y' \end{pmatrix} Z_2 \begin{pmatrix} x & y \\ x'' & y'' \end{pmatrix} + \lambda Z_2 \begin{pmatrix} x & x \\ x'' & y'' \end{pmatrix} Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \\ = \delta(x-y') Z_2 \begin{pmatrix} x' & y \\ x'' & y'' \end{pmatrix} + \delta(x-y'') Z_2 \begin{pmatrix} x'' & y \\ x' & y' \end{pmatrix}. \end{aligned} \quad (25)$$

The system of three equations (21), (23) and (25) for two functions  $\Delta$  and  $Z_2$  are overfull and seemingly has not physically meaningful solutions. However, as it will be show below, equation (25) does not play a role for the construction of the modified mean-field expansion. Really, for this construction we define the functional  $\bar{G}$  as

$$Z = Z_{part}(\eta) + \log \bar{G}. \quad (26)$$

Taking into account equations (21) and (23) (but not (25)!), we obtain the SDE for the functional  $\bar{G}$ :

$$\begin{aligned} (m^2 - \partial^2) \frac{\delta \bar{G}}{\delta \eta} - \lambda \frac{\delta^2 \bar{G}}{\delta \eta \delta \eta} + \lambda \frac{\delta \bar{G}}{\delta \eta} \Delta + \eta \cdot \frac{\delta \bar{G}}{\delta \eta} - \lambda \frac{\delta \bar{G}}{\delta \eta} Z_2 \cdot \eta - \lambda Z_2 \cdot \eta \frac{\delta \bar{G}}{\delta \eta} = \\ = \bar{G} \left[ \lambda Z_2 \cdot \eta Z_2 \cdot \eta - \eta \cdot Z_2 \cdot \eta \right] \end{aligned} \quad (27)$$

(in operator notation; the central dot is an integration). If we require also to satisfy equation (25), then rhs of equation (27) is zero, and we obtain the situation, which in the theory of ordinary differential equations is named as "the reduction of order": instead of the second-order linear equation (3), we have a first-order linear equation for  $\delta \bar{G} / \delta \eta$ . But the construction of the mean-field expansion for  $\bar{G}$ , which is similar to the mean-field expansion for  $G$  of section 2, absolutely has not need of such reduction. For this reason we shall not require a satisfaction of equation (25).

To construct the mean-field expansion for  $\bar{G}$  we consider as a leading-order approximation (in full correspondence with the principle of the construction of mean-field expansion of section 2) the terms of equation (27), which do not contain the source  $\eta$  manifestly, i.e. the leading-order equation is

$$(m^2 - \partial_x^2) \frac{\delta \bar{G}_0}{\delta \eta(y, x)} - \lambda \frac{\delta^2 \bar{G}_0}{\delta \eta(x, x) \delta \eta(y, x)} + \lambda \frac{\delta \bar{G}_0}{\delta \eta(x, x)} \Delta(x-y) = 0 \quad (28)$$

All other terms (including the rhs of (27)) are considered as the perturbation.



This equation also has a linear exponent as a solution. Then, in correspondence with section 2, we have the iteration scheme

$$\bar{G} = \bar{G}^{(0)} + \bar{G}^{(1)} + \dots + \bar{G}^{(n)} + \dots,$$

and the equation for  $\bar{G}^{(n)}$  is

$$\begin{aligned} (m^2 - \partial^2) \frac{\delta \bar{G}_n}{\delta \eta} - \lambda \frac{\delta^2 \bar{G}_n}{\delta \eta \delta \eta} + \lambda \frac{\delta \bar{G}_n}{\delta \eta} \Delta + \left\{ \eta \cdot \frac{\delta \bar{G}_{n-1}}{\delta \eta} - \lambda \frac{\delta \bar{G}_{n-1}}{\delta \eta} Z_2 \cdot \eta - \lambda Z_2 \cdot \eta \frac{\delta \bar{G}_{n-1}}{\delta \eta} \right\} = \\ = \left[ Z_2 \cdot \eta Z_2 \cdot \eta - \eta \cdot Z_2 \cdot \eta \right] \bar{G}_{n-2} \end{aligned}$$

The general solution of this equation is the functional  $\bar{G}^{(n)} = \bar{P}^{(n)} \bar{G}^{(0)}$ , where  $\bar{P}^{(n)}$  is a polynomial on  $\eta$ . At  $n$ th step of this iteration scheme, we have a closed system of linear integral equations, and therefore this scheme is much less complicated in comparison to the scheme of preceding section. Equations (21) and (23) are the basic approximation for this expansion.

## 5 The renormalized equations of the two-particle approximation

Equations (21) and (23) are the system of non-linear equations for the functions  $\Delta$  and  $Z_2$ . In equation (23), the two-particle function  $Z_2$  can be considered as a functional of  $\Delta$ , and the "solution" of this equation can be easily found:

$$\begin{aligned} Z_2 \left( \begin{array}{cc} x & y \\ x' & y' \end{array} \right) = \Delta_c(x - y') \Delta(x' - y) - \\ - \int dx_1 dx_2 \Delta_c(x - x_1) \Delta(x' - x_2) f(x_1 - x_2) \Delta(x_1 - y) \Delta_c(x_2 - y'). \end{aligned} \quad (29)$$

Here, function  $f(p)$  in the momentum space is a solution of the equation

$$\frac{1}{f(p)} = \frac{1}{\lambda} + \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p + q) \Delta(q), \quad (30)$$

where

$$\Delta_c(p) = \frac{1}{m^2 + p^2}. \quad (31)$$

Taking into account equations (29) and (30), we obtain for  $\Delta$  the following equation in the momentum space

$$(m^2 + p^2) \Delta(p) = 1 - \Delta(p) \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p - q) f(q). \quad (32)$$

Note that a very crude approximation for equation (32) is

$$(m^2 + p^2)\Delta(p) \approx 1 \implies \Delta(p) \approx \Delta_c(p). \quad (33)$$

Within this approximation we obtain as a solution of equation (30) the mean-field amplitude of equation (8). So the mean-field approximation and the equivalent leading-order  $1/N$ -expansion are contained in the two-particle approximation as an approximate solution.

The renormalization of equations (30) and (32) can be performed in correspondence with the general recipe of section 2 by introducing counter-term Lagrangian (11) and full Lagrangian (12). In terms of the bare quantities the bare propagator is

$$\Delta_b^{-1}(p^2) = m_b^2 + \lambda_b \Delta_b(x=0) + p^2 + \Sigma_b(p^2), \quad (34)$$

where

$$\Sigma_b(p^2) = \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p-q) f_b(q) \quad (35)$$

is the bare mass operator.

The normalization conditions (14) and (15) for the renormalized propagator  $\Delta = z^{-1} \Delta_b$  define the mass-renormalization counter-term  $\delta m^2$  and the field-renormalization constant

$$z = (1 + \Sigma'_b(0))^{-1}. \quad (36)$$

Then the equation for the renormalized propagator is

$$(m^2 + p^2)\Delta(p^2) = 1 - \Delta(p^2)\Sigma_r(p^2), \quad (37)$$

where

$$\Sigma_r(p^2) = z[\Sigma_b(p^2) - \Sigma_b(0) - p^2 \Sigma'_b(0)] \quad (38)$$

is the renormalized mass operator.

The equation for  $f_b$  is

$$\frac{1}{f_b(p)} = \frac{1}{\lambda_b} + L_b(p), \quad (39)$$

where

$$L_b(p) = \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p+q) \Delta_b(q) = z \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p+q) \Delta(q) \quad (40)$$

is the bare loop operator.

The renormalized amplitude  $F$  is defined as an amputation of the connected part  $Z_2^{con}$  of the renormalised two-particle function  $Z_2$ :

$$Z_2^{con} = -\Delta \cdot \Delta \cdot F \cdot \Delta \cdot \Delta.$$

In correspondence with the bare version of equation (29)

$$Z_2^{con} = z^{-2} Z_{2b}^{con} = -z^{-2} \Delta_c \cdot \Delta_b \cdot f_b \cdot \Delta_b \cdot \Delta_c = -\Delta_c \cdot \Delta \cdot f_b \cdot \Delta \cdot \Delta_c$$

and we have (in momentum space):

$$F \begin{pmatrix} p_x & p_y \\ p'_x & p'_y \end{pmatrix} = \Delta^{-1}(p_x) \Delta_c(p_x) f_b(p_x + p_y) \Delta_c(p'_y) \Delta^{-1}(p'_y).$$

The renormalized coupling  $\lambda$  is defined as

$$F|_{p_i=0} = \lambda,$$

and due to the normalization condition (14) for the propagator we have

$$f_b(0) = \lambda. \quad (41)$$

This normalization condition together with equation (39) defines the coupling-renormalization constant  $z_\lambda$  and the renormalized equation for  $f \equiv f_b$ :

$$\frac{1}{f(p^2)} = \frac{1}{\lambda} + L_r(p^2), \quad (42)$$

where

$$L_r(p^2) = L_b(p^2) - L_b(0) \quad (43)$$

is the renormalized loop operator.

Equations (37) and (42) are the system of nonlinear integral equations for the propagator and the amplitude. This system with taking into account the normalization conditions (41) and (14)–(15) can be solved by the expansion in the vicinity of the point  $p = 0$ . Such solution, however, is not interesting because it is some part of the usual perturbation theory over the coupling  $\lambda$ .

Much more interesting problem is to look for the asymptotic behavior of the solution at large momenta. In the large-momenta region, an essential technical simplification is possible, namely, one can replace in integrals (35) and (40) the function  $\Delta_c$  (see (31)) by massless function  $1/p^2$ :

$$\int \frac{d^4 q}{(2\pi)^4} \Phi(q^2) \Delta_c(p - q) \Rightarrow \int \frac{d^4 q}{(2\pi)^4} \frac{\Phi(q^2)}{(p - q)^2}. \quad (44)$$

Then it is possible to use the well-known formula

$$\int \frac{d^4 q}{(2\pi)^4} \frac{\Phi(q^2)}{(p - q)^2} = \frac{1}{16\pi^2} \left[ \frac{1}{p^2} \int_0^{p^2} \Phi(q^2) q^2 dq^2 + \int_{p^2}^\infty \Phi(q^2) dq^2 \right]. \quad (45)$$

The approximation (44) is quite usual in investigations in the deep-Euclidean region, and formula (45) highly enables the calculations and, as a major point, permits us to go from integral equations to differential ones (see below).

As an example of using formula (45), we calculate the field-renormalization constant  $z$ . Due to the normalization condition (41) we have  $f \rightarrow \lambda$  at  $p \rightarrow 0$ . Consequently, with the application of formula (45) we obtain

$$\Sigma'_b(0) = -\frac{1}{2} \cdot \frac{\lambda}{16\pi^2}, \quad (46)$$

and, taking into account (36),

$$z = \frac{2}{2 - \lambda/16\pi^2}. \quad (47)$$

Due to the positivity of  $z$ , this formula implies the limitation  $\lambda < 32\pi^2 \approx 320$  on all following calculations.

Further essential simplification of the system of equations for  $f$  and  $\Delta$  consists in replacing of equation (37) by the approximate relation

$$(m^2 + p^2)\Delta(p^2) \approx 1 - \Delta_c(p^2)\Sigma_r(p^2). \quad (48)$$

This approximation enables us "to unleash" the system, keeping at the time the non-linearity, i.e., to obtain the non-linear integral equation for  $f$ , which does not contain  $\Delta$ . This approximation, of course, is less substantiated and should be considered as an iteration of the initial equations. Some grounds for this approximation can be obtained *a posteriori* if the deep-Euclidean behavior of the propagator  $\Delta$  does not essentially differ from the behavior of the free propagator.

Approximation (48) defines the equation for  $f$ , which will be a main object of the following consideration. For the future convenience we introduce new quantities  $y$ ,  $g$  and dimensionless variable  $t$  as

$$y = \frac{z}{16\pi^2} f, \quad g = \frac{z}{16\pi^2} \lambda, \quad t = \frac{p^2}{m^2}. \quad (49)$$

The normalization condition for  $y(t)$  is

$$y(0) = g. \quad (50)$$

After a simple calculation, we obtain the equation for  $y(t)$  in the form

$$\frac{1}{y} = \frac{1}{g} + l(t) + \int_0^t dt_1 K(t|t_1) y(t_1), \quad (51)$$

where

$$l(t) = \left(\frac{g}{2} - 1\right) \log(1+t) + (1-g)\left(1 - \frac{1}{t} \log(1+t)\right), \quad (52)$$

and the kernel  $K$  is

$$K(t|t_1) = \frac{t_1}{t} - 1 + \frac{1}{t} \log \frac{1+t}{1+t_1} + t_1 \log \frac{t(1+t_1)}{t_1(1+t)}. \quad (53)$$

Integral equation (51) is the non-linear Volterra equation and can be reduced to the non-linear differential equation of the fourth order

$$\frac{d^2}{dt^2} \left[ t(t+1)^2 \frac{d^2}{dt^2} \left( \frac{t}{y} \right) \right] = g - 2 - y. \quad (54)$$

It worth noting that the differential equation (54) will be used only for a suggestion the form of the asymptotic solution of the integral equation (51).

## 6 Asymptotic solution for the amplitude

As it easy to see, the differential equation (54) at  $g \neq 2$  has the exact solution

$$y_{exact} = g - 2. \quad (55)$$

This solution is not, however, the exact solution of the integral equation (51), but the leading term of the asymptotic expansion at  $t \rightarrow \infty$ . Indeed, a simple calculation demonstrates that after the substitution of (55) into (51) the leading increasing logarithmic term is cancelled. Hence, the asymptotic solution of the integral equation (51) at  $g \neq 2$  has the form

$$y = g - 2 + \varphi(t), \quad (56)$$

where  $\varphi(t) \rightarrow 0$  at  $t \rightarrow \infty$ . The function  $\varphi(t)$  at large  $t$  satisfies to the linear differential equation

$$\frac{d^2}{dt^2} \left[ t^3 \frac{d^2}{dt^2} (t\varphi) \right] = (g - 2)^2 \varphi. \quad (57)$$

This equation suggests the form of next-to-the-leading asymptotic term  $\varphi$ . The solutions of this Euler-type equation are  $t^{\alpha-1}$ , where

$$\alpha^2(\alpha^2 - 1) = (g - 2)^2. \quad (58)$$

This characteristic equation has two real and two imaginary roots. The first real root is greater than 1, and corresponds to an increasing solution. The second real root is less than -1, and corresponds to a fast-decreasing solution. The imaginary roots correspond to decreasing as  $t^{-1}$  and oscillating solutions. Hence, just these roots define the next-to-the-leading asymptotic term of the solution of integral equation (51).

For further calculation it is convenient to go to the variable

$$x = \log(1 + t) \quad (59)$$

In terms of this variable the next-to-the-leading asymptotic term is

$$\varphi = e^{-x} (A \cos \omega x - B \sin \omega x), \quad (60)$$

where

$$\omega = \sqrt{\sqrt{(g-2)^2 + \frac{1}{4}} - \frac{1}{2}}. \quad (61)$$

Substituting (56) and (60) into the integral equation (51) gives us at  $t \rightarrow \infty$  the relation

$$\frac{1}{g-2} = \frac{1}{g} - 1 - a(\omega)A + \omega b(\omega)B, \quad (62)$$

where

$$a(\omega) = \sum_{k=1}^{\infty} \frac{1}{(k+1)(k^2 + \omega^2)}, \quad b(\omega) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k^2 + \omega^2)}. \quad (63)$$

These sums can also be expressed in terms of the special functions (see, e.g. [13]).

Due to the normalization condition (50) we have  $A = 2$ , and equation (62) defines the quantity  $B(g)$ .

## 7 Asymptotic behavior of the amplitude

At  $t \rightarrow \infty$  we have  $y \rightarrow g-2$ . At  $g < 2$  such behavior contradicts to continuity. If the function  $y$  is continuous, then at  $g < 2$   $y(t_0) = 0$  in some point  $t_0 \in (0; +\infty)$  (since  $y$  changes the sign). But in the case the integral equation (51) is not fulfilled in the point  $t_0$ . Consequently the function  $y$  is singular in some point of pre-asymptotic region (Landau pole, or something similar), and at  $g < 2$  this model is inconsistent beyond the region of small momenta.

Hence, the region of applicability of this model is  $g > 2$ , or

$$\lambda_{cr} < \lambda < 2\lambda_{cr}, \quad (64)$$

where

$$\lambda_{cr} = 16\pi^2 \cong 160. \quad (65)$$

The first two terms of asymptotic expansion in this interval of  $\lambda$  at large  $t = \frac{p^2}{m^2}$  are given as follows

$$f \simeq 2(\lambda - \lambda_{cr}) + \frac{2\lambda_{cr} - \lambda}{t} \left( \cos(\omega \log t) - \frac{B(\lambda)}{2} \sin(\omega \log t) \right), \quad (66)$$

where

$$\omega(\lambda) = \sqrt{\sqrt{\left(\frac{4(\lambda - \lambda_{cr})}{2\lambda_{cr} - \lambda}\right)^2 + \frac{1}{4}} - \frac{1}{2}}, \quad (67)$$

and

$$B(\lambda) = \frac{1}{\omega b(\omega)} \left[ \frac{(2\lambda_{cr} - \lambda)^2}{4\lambda(\lambda - \lambda_{cr})} + 1 + 2a(\omega) \right], \quad (68)$$

At  $\lambda \rightarrow \lambda_{cr}$

$$f \simeq 2(\lambda - \lambda_{cr}) + \frac{C}{(\lambda - \lambda_{cr})^2} \frac{1}{t} \sin(\omega \log t), \quad (69)$$

where  $C = (2\pi)^6/b(0)$ , i.e., in the vicinity of the critical point  $\lambda_{cr}$  the amplitude of oscillations tends to infinity and the solution is destroyed.

On the other end of the interval at  $\lambda \rightarrow 2\lambda_{cr}$

$$f \simeq 2\lambda_{cr} - 4\pi\sqrt{2\lambda_{cr} - \lambda} \frac{1}{t} \sin(\omega \log t), \quad (70)$$

i.e.,  $f(t) \rightarrow f(0)$  – the amplitude at  $t \rightarrow \infty$  tends to the initial value of the point  $t = 0$ . In other words, at  $\lambda = 2\lambda_{cr}$  the asymptotic reconstruction of the amplitude occurred.

At  $\lambda > 2\lambda_{cr}$  the method of solution is, strictly speaking, cannot be applied due to the negative value of the field-renormalization constant  $z$ , but if one formally assumes such values, the result of equation (66) can be continued to the region of strong coupling.

## 8 Conclusions

The most interesting feature of the result obtained is, of course, the existence of the critical point  $\lambda_{cr}$ , which divides the whole set of coupling values in two region: the weak-coupling region of inconsistency and the strong-coupling region of non-trivial self-consistent behavior.<sup>4</sup> (The existence of the second specific point seems to be an artefact of the calculation scheme.) This division of the coupling values cannot be indicated by the expansions tightly connected with perturbation theory (e.g., the renormalization-group summation and the  $1/N$ -expansion). On the other hand, the strong-coupling limit is also not susceptible to the existence of such division, since in the framework of the Schwinger-Dyson formalism the strong-coupling expansion requires for the physically-acceptable interpretation some subsidiary summation [14, 15] based on unknown grounds.

## Aknowlegements

Author is grateful to the participants of IHEP Theory Division Seminar for useful discussion.

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<sup>4</sup>Note that a singular behavior on parameters for the theory of four-dimensional scalar field was marked before. Some years ago, using the local potential approach to the Wilson renormalization group, Halpern and Huang [16] have found a class of non-trivial interactions of a scalar field in four dimensions. As pointed by Morris [17], such interactions correspond to singular effective potential. The connection of this fact with our result is, however, unclear due to very different methods of investigation.

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